DEVELOPMENT OF THE THEORY OF HYPERGEOMETRIC FUNCTIONS IN CONNECTION WITH PROBLEMS ON ELASTIC EQUILIBRIUM OF PLATES AND SHELLS

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A number of problems on elastic equilibrium of circular plates of variable thickness and of shells of revolution, can be reduced to integration of hypergeometric equations (Either Fuchsian or degenerate). Some particular solutions of these equations are logarithmic. In connection with the construction of solutions of such problems described by second order hypergeometric equations, the author introduced in [1] a hypergeometric function $\Phi(a, b; c; z)$ of the second kind and established its fundamental functional relations analogous to those available for the hypergeometric function $F(a, b; c; z)$.

Functional relations of $\Phi(a, b; c; z)$ have found applications in the theory of circular plates of variable thickness, conical shells of linearly variable thickness [2, 3 and 4] e.a.

At a later date this function was investigated by Nörlund in [5] who also introduced another function

$$\Psi(a, b; c; z) = \Phi(a, b; c; z) + [\psi(a) + \psi(b) - \psi(c) - -\psi(1)] F(a, b; c; z)$$
$$(\psi(a) = d \ln \Gamma(a)/da) \qquad (0.1)$$

and in his paper functions Φ and Ψ were denoted by G and g respectively. Here we have adopted the notation $\Psi(a, b; c; z)$ as logically related to $\Psi(a; c; z)$ which was used in a logarithmic solution of a degenerate hypergeometric equation given in [6], p. 248.

Functional relations for $\Psi(a, b; c; z)$ which we shall also call a hypergeometric function of the second kind, are simpler than those for $\Phi(a, b; c; z)$.

Fairly recently, various authors [3, 4 and 7 to 10] introduced generalized hypergeometric functions of the second kind and investigated some of their properties in connection with problems on asymmetric deformation of circular plates of variable thickness and of hollow conical shells of revolution.

We shall denote a generalized hypergeometric function of the second kind containing a term with $\ln z$, by ${}_p\Phi_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$. In analogy with the functions (0.1) we introduce, for this case, a function

$${}_p\Psi_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = {}_p\Phi_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) +$$
$$+ [\psi(\alpha_1) + \ldots + \psi(\alpha_p) - \psi(\beta_1) - \ldots - \psi(\beta_q) - \psi(1)]\, {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots \beta_q; z) \qquad (0.2)$$

Generalized hypergeometric function of the second kind containing terms with $\ln z$ and $(\ln z)^2$, terms with $\ln z$, $(\ln z)^2$ and $(\ln z)^3$ etc. shall be denoted by ${}_p\Phi_q^{(2)}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$, ${}_p\Phi_q^{(3)}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$ etc.

The purpose of the present paper is to generalize the results obtained by the author and his collaborators on the properties of hypergeometric functions of the second kind and to illustrate it with examples of their effective use in problems on the state of stress in plates and shells.

1. Some fundamental properties of hypergeometric functions. The

basic system of particular solutions of a hypergeometric Eq.

$$z(1-z)\frac{d^2W}{dz^2}+[c-(a+b+1)z]\frac{dW}{dz}-abW=0 \tag{1.1}$$

near $z=0$ when $c=1+m(m=0, 1,\ldots)$ and $a, b\neq 1, 2,\ldots, m$, is given by functions

$$F(a, b; c; z)=1+\sum_{n=1}^{\infty}\frac{[a]_n[b]_n}{[c]_n}\frac{z^n}{n!} \tag{1.2}$$

$$\Phi(a, b; c; z)=\ln z\,F(a, b; c; z)+\sum_{n=1}^{c-1}(-1)^{n-1}\frac{(n-1)!\,[c-n]_n}{[a-n]_n[b-n]_n}z^{-n}+$$

$$+\sum_{n=1}^{\infty}\frac{[a]_n[b]_n}{[c]_n}\frac{z^n}{n!}\sum_{s=1}^{n}\left(\frac{1}{a+s-1}+\frac{1}{b+s-1}-\frac{1}{c+s-1}-\frac{1}{s}\right) \tag{1.3}$$

$$([a]_n=a(a+1)\ldots(a+n-1),\ [a]_0=1)$$

The linear combination (0.1) of particular solutions (1.2) and (1.3) should also be considered as a logarithmic solution of (1.1).

Functions $\Phi(a, b; c; z)$ and $\Psi(a, b; c; z)$ are single valued analytic functions of z in the region $|\arg z|<\pi$ with a cut along the real axis from $-\infty$ to 0. They can be represented within a unit circle $|z|<1$ by expressions of the type (1.3) and (0.1), containing convergent series. Outside the circle of convergence they can be found by the method of analytic continuation of indicated expressions.

Expression (1.3) becomes meaningless for $\Phi(a, b; c; z)$ when one of the parameters a and b assumes one of the following values: $1, 2,\ldots, c-1$, and (0.1) becomes meaningless for $\Psi(a, b; c; z)$ when, in addition, a or b becomes zero or assumes a negative integral value. In these cases, particular solutions of (1.1) will be rational functions. We shall not consider them in this paper and we shall therefore assume that $a, b\neq c-1,\ldots, 1, 0, -1,\ldots$

In [1] we gave for the function $\Phi(a, b; c; z)$ the differentiation and integration formulas, a transformation fomula, dependence of functions on their arguments z and $1-z$ as well as some recurrent relations for functions whose parameters a, b and c differed in values by whole numbers and a full system of recurrent relations between contiguous functions.

Using (0.1) to pass to functions $\Psi(a, b; c; z)$ in the formulas and relations just mentioned, we can see that most of them become simplified in the process. Below we give the basic functional relations for $\Psi(a, b; c; z)$ which have been found useful in constructing solutions of problems on axisymmetric deformation of circular plates of variable thickness and of shells of revolution (conical shells of linearly varying thickness and spherical shells of constant thickness).

A differentiation Formula

$$\frac{d}{dz}\Psi(a, b; c; z)=\frac{ab}{c}\Psi(a+1, b+1; c+1; z) \tag{1.4}$$

a transformation Fomula when $a+b=0, \pm 1, \ldots$

$$\Psi(a, b; c; z)=(1-z)^{c-a-b}\,\Psi(c-a, c-b; c; z) \tag{1.5}$$

and a set of 15 relations between contiguous functions

$$\begin{aligned}
&1.\ [c-2a-(b-a)z]\Psi+a(1-z)\Psi(a+1)-(c-a)\Psi(a-1)=0\\
&2.\ (b-a)\Psi+a\Psi(a+1)-b\Psi(b+1)=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&14.\ [b-1-(c-a-1)z]\Psi+\\
&\qquad+(c-b)\Psi(b-1)-(c-1)(1-z)\Psi(c-1)=0\\
&15.\ c[c-1-(2c-a-b-1)z]\Psi+\\
&\qquad+(c-a)(c-b)z\Psi(c+1)-c(c-1)(1-z)\Psi(c-1)=0
\end{aligned} \tag{1.6}$$

$$\Psi=\Psi(a, b; c; z),\quad \Psi(a\pm 1)=\Psi(a\pm 1, b; c; z)$$

$$\Psi(b\pm 1)=\Psi(a, b\pm 1; c; z),\ \Psi(c\pm 1)=\Psi(a, b; c\pm 1; z)$$

which outwardly coincides with 15 Gaussian relations for contiguous functions $F(a, b; c; z)$ (see [11], p. 130).

Linear dependence of hypergeometric functions on z and $1-z$ when $a+b=0; \pm 1, \ldots$ is given by

$$\Psi(a, b; c; z) = (-1)^c \frac{\Gamma(1-a)\Gamma(1-b)(c-1)!}{\Gamma(c-a-b+1)}(1-z)^{c-a-b} \times$$
$$\times F(c-a, c-b; c-a-b+1; 1-z)$$

$$(c = 1+m;\ m = 0, 1, \ldots;\ a, b \neq 1, 2, \ldots;\ |\arg z| < \pi, |\arg(1-z)| < \pi) \tag{1.7}$$

Formulas (1.4) and (1.5) also coincide with the corresponding formulas for $F(a, b; c; z)$. We should note that the condition $a+b=0, \pm 1, \ldots$ holds for (1.5) and (1.7) in problems under consideration.

Applying the relations (1.6) once again, we can derive linear relations between $\Psi(a, b; c; z)$ and another two functions of the type $\Psi(a+l, b+m; c+n; z)$ where l, m and n are integers and in which coefficients are polynomials in z. The latter relations also outwardly coincide with corresponding recurrent formulas for $F(a, b; c; z)$. Integral representation and a formula for analytic continuation of $\Psi(a, b; c; z)$ are obtained as particular cases of (4.7) and (4.8) for ${}_p\Psi_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$.

2. Logarithmic solutions of hypergeometric equations of higher order. A hypergeometric equation of the $(q+1)$th order has the form

$$\left\{z\frac{d}{dz}\left(z\frac{d}{dz}+\beta_1-1\right)\ldots\left(z\frac{d}{dz}+\beta_q-1\right)-\right.$$
$$\left.-z\left(z\frac{d}{dz}+\alpha_1\right)\ldots\left(z\frac{d}{dz}+\alpha_p\right)\right\}W = 0 \tag{2.1}$$

where $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are arbitrary complex parameters while p and q are positive integers $(p \leqslant q+1)$. When $p = q+1$, then Eq. (2.1) is Fuchsian and has three essential singularities $z=0$, $z=1$ and $z=\infty$. When $p \leqslant q$, then (2.1) becomes a degenerate hypergeometric equation of the $(q+1)$th order with one essential singularity $z=0$ and one removable singularity $z=\infty$. If none of the parameters β_r $(r = 1, 2, \ldots, q)$ and none of the differences $\beta_s - \beta_t$ $(s \neq t;\ s, t = 1, 2, \ldots, q)$ are equal to zero or to an integer, then the basic system of particular solutions of (2.1) has the form [12 and 13]

$$W_1 = {}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z) \tag{2.2}$$

$$W_{n+1} = z^{1-\beta_n}{}_pF_q(1+\alpha_1-\beta_n, \ldots, 1+\alpha_p-\beta_n;\ 2-\beta_n$$
$$1+\beta_1-\beta_n{}^*, \ldots, 1+\beta_q-\beta_n;\ z) \quad (n = 1, 2, \ldots, q)$$

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = 1 + \sum_{n=1}^{\infty} \frac{[\alpha_1]_n \ldots [\alpha_p]_n}{[\beta_1]_n \ldots [\beta_q]_n} \frac{z^n}{n!} \tag{2.3}$$

where ${}_pF_q$ is a generalized hypergeometric function and an asterisk means that the parameter $1+\beta_s-\beta_t$ is dropped when $s = t$.

Series (2.3) converges for all finite z when $p \leqslant q$ and for $|z| < 1$ when $p = q+1$. When some of the parameters β_s $(s = 1, 2, \ldots, q)$ assume positive integral values, then (2.1) has a corresponding number of logarithmic solutions, which can be obtained in terms of various forms of generalized hypergeometric functions of the second kind.

If e.g. $\beta_1 = 1+m$ $(m = 0, 1, \ldots)$, then, using either the method of limits [2] or the Frobenius method [14], we can obtain the following second particular solution

$$W_2 = {}_p\Phi_q(\alpha_1, \ldots, \alpha_p;\ 1+m, \beta_2, \ldots, \beta_q;\ z) =$$

$$= \ln z\, {}_pF_q(\alpha_1, \ldots, \alpha_p;\ 1+m, \beta_2, \ldots, \beta_q;\ z) +$$

$$+ \sum_{n=1}^{m} (-1)^{n-1} \frac{(n-1)!\,[1+m-n]_n [\beta_2-n]_n \ldots [\beta_q-n]_n}{[\alpha_1-n]_n \ldots [\alpha_p-n]_n} z^{-n} +$$

$$+ \sum_{n=1}^{\infty} \frac{[\alpha_1]_n \ldots [\alpha_p]_n}{[1+m]_n [\beta_2]_n \ldots [\beta_q]_n} \frac{z^n}{n!} \sum_{s=1}^{n} \left(\frac{1}{\alpha_1+s-1} + \ldots + \frac{1}{\alpha_p+s-1} -\right.$$

$$-\frac{1}{m+s}-\frac{1}{\beta_2+s-1}-\ldots-\frac{1}{\beta_q+s-1}-\frac{1}{s}\Bigg) \tag{2.4}$$

$$(\alpha_r \neq 1, \ldots, m;\ r=1, \ldots, p)$$

When $\beta_1 = 1 + m$ and $\beta_2 = 1 + k$ $(m, k = 0, 1, \ldots; m \leqslant k)$, a third particular solution is obtained by the Frobenius method in the form [10]

$$W_3 = {}_p\Phi_q^{(2)}(\alpha_1, \ldots, \alpha_p;\ 1+m, 1+k, \ldots, \beta_q;\ z) =$$

$$= -(\ln z)^2\ {}_pF_q(\alpha_1, \ldots, \alpha_p;\ 1+m,\ 1+k, \ldots, \beta_q;\ z) +$$

$$+ 2\ln z\ {}_p\Phi_q(\alpha_1, \ldots, \alpha_p;\ 1+m,\ 1+k, \ldots, \beta_q;\ z) +$$

$$+ 2\sum_{n=1}^{m}(-1)^{n-1}\frac{(n-1)!\,[1+m-n]_n\,[1+k-n]_n\cdots[\beta_q-n]_n}{[\alpha_1-n]_n\cdots[\alpha_p-n]_n}A_{-n}z^{-n} +$$

$$+ 2(-1)^m m!\sum_{n=m+1}^{k}\frac{(n-1)!\,(n-m-1)!\,[1+k-n]_n\cdots[\beta_q-n]_n}{[\alpha_1-n]_n\cdots[\alpha_p-n]_n}z^{-n} +$$

$$+\sum_{n=1}^{\infty}\frac{[\alpha_1]_n\cdots[\alpha_p]_n}{[1+m]_n\,[1+k]_n\cdots[\beta_q]_n}(A_n^2 - B_n)\frac{z^n}{n!} \tag{2.5}$$

$$(\alpha_r \neq 1, \ldots, k;\ r=1, \ldots, p)$$

$$A_n = \sum_{s=1}^{n}\left(\frac{1}{\alpha_1+s-1}+\cdots+\frac{1}{\alpha_p+s-1}-\frac{1}{m+s}-\cdots-\frac{1}{\beta_q+s-1}-\frac{1}{s}\right)$$

$$B_n = \sum_{s=1}^{n}\left[\frac{1}{(\alpha_1+s-1)^2}+\cdots+\frac{1}{(\alpha_p+s-1)^2}-\frac{1}{(m+s)^2}-\cdots-\frac{1}{(\beta_q+s-1)^2}-\frac{1}{s^2}\right]$$

$$A_{-n} = \sum_{s=1}^{n}{}^{*}\left(\frac{1}{m-n+s}+\cdots+\frac{1}{\beta_q-n+s-1}-\frac{1}{\alpha_1-n+s-1}-\cdots\right.$$

$$\left.\cdots-\frac{1}{\alpha_p-n+s-1}-\frac{1}{s-1}\right) \tag{2.6}$$

Here an asterisk means that the term $1/(s-1)$ is dropped when $s = 1$.

Using the method indicated above we can obtain logarithmic solutions of (2.1) in the case when three or more parameters $\beta_s (s = 1, 2, \ldots, q)$ are equal to positive integers. These solutions are given in terms of generalized hypergeometric functions of the second kind, containing $\ln z$ in the third and higher powers. Thus if $\beta_1 = 1 + m$, $\beta_2 = 1 + k$ and $\beta_3 = 1 + l$ $(m, k, l = 0, 1, \ldots; m \leqslant k \leqslant l)$, then we have a fourth solution in the form

$$W_4 = {}_p\Phi_q^{(3)}(\alpha_1, \ldots, \alpha_p;\ 1+m,\ 1+k,\ 1+l, \ldots, \beta_q;\ z) =$$

$$= (\ln z)^3\ {}_pF_q(\alpha_1, \ldots, \alpha_p;\ 1+m,\ 1+k,\ 1+l, \ldots, \beta_q;\ z) -$$

$$-3(\ln z)^2\ {}_p\Phi_q(\alpha_1, \ldots, \alpha_p;\ 1+m,\ 1+k,\ 1+l, \ldots, \beta_q;\ z) +$$

$$+3\ln z\ {}_p\Phi_q^{(2)}(\alpha_1, \ldots, \alpha_p;\ 1+m, 1+k, 1+l, \ldots, \beta_q;\ z) +$$

$$+3\sum_{n=1}^{m}(-1)^{n-1}\frac{(n-1)!\,[1+m-n]_n\cdots[\beta_q-n]_n}{[\alpha_1-n]_n\cdots[\alpha_p-n]_n}(A_{-n}^2+B_{-n})z^{-n} +$$

$$+6(-1)^m m!\sum_{n=m+1}^{k}\frac{(n-1)!\,(n-m-1)!\,[1+k-n]\cdots[\beta_q-n]_n}{[\alpha_1-n]_n\cdots[\alpha_p-n]_n}C_{-n}z^{-n}+6(-1)^{m+k}\times$$

$$\times m!\,k!\sum_{n=k+1}^{l}(-1)^{n-1}\frac{(n-1)!\,(n-m-1)!\,(n-k-1)!\,[1+l-n]_n\cdots[\beta_q-n]_n}{[\alpha_1-n]_n\cdots[\alpha_p-n]_n}z^{-n}+$$

$$+\sum_{n=1}^{\infty}\frac{[\alpha_1]_n\cdots[\alpha_p]_n}{[1+m]_n\,[1+k]_n\,[1+l]_n\cdots[\beta_q]_n}\,(A_n^3-3A_nB_n+2C_n)\frac{z^n}{n!}$$

$$(\alpha_r\neq 1,2,\ldots,l;\ r=1,\ldots,p) \qquad (2.7)$$

where

$$C_n=\sum_{s=1}^{n}\left[\frac{1}{(\alpha_1+s-1)^3}+\cdots+\frac{1}{(\alpha_p+s-1)^3}-\frac{1}{(m+s)^3}-\cdots-\frac{1}{(\beta_q+s-1)^3}-\frac{1}{s^3}\right]$$

$$B_{-n}=\sum_{s=1}^{n}{}^{*}\left[\frac{1}{(\alpha_1-n+s-1)^2}+\cdots+\frac{1}{(\alpha_p-n+s-1)^2}-\frac{1}{(m-n+s)^2}-\cdots\right.$$

$$\left.\cdots-\frac{1}{(\beta_q-n+s-1)^2}-\frac{1}{(s-1)^2}\right]$$

$$C_{-n}=\psi(1+m)-\psi(n-m)+\sum_{s=1}^{n}{}^{*}\left(\frac{1}{k-n+s}+\cdots+\frac{1}{\beta_q-n+s-1}-\right.$$

$$\left.-\frac{1}{\alpha_1-n+s-1}-\cdots-\frac{1}{\alpha_p-n+s-1}-\frac{1}{s-1}\right) \qquad (2.8)$$

Generalized hypergeometric functions of the second kind (2.4), (2.5) and (2.7) are symmetric in parameters $\alpha_r (r=1,\ldots,p)$ and in parameters β_s with nonintegral (fractional) values. Parameters β_s with integral values are written in a nondecreasing sequence and they define the number of terms with negative powers in the series (2.4), (2.5) and (2.7).

If one or more parameters $\alpha_r (r=1,\ldots,p)$ are equal to one of the values $1,\ldots,l$ when $\beta_1=1+m$, $\beta_2=1+k$ and $\beta_3=1+l$ $(m,k,l=0,1,\ldots;\ m\leqslant k\leqslant l)$, then the number of logarithmic solutions diminishes and some of the particular solutions degenerate into elementary functions. The construction of solutions for various cases of parameters $\alpha_r (r=1,\ldots,p)$ and $\beta_s (s=1,\ldots,q)$ possessing integral values, shall not be considered here.

3. Logarithmic solutions of an inhomogeneous hypergeometric equation of higher order. The particular solution of such an equation

$$\left\{z\frac{d}{dz}\left(z\frac{d}{dz}+\beta_1-1\right)\cdots\left(z\frac{d}{dz}+\beta_q-1\right)-\right.$$

$$\left.-z\left(z\frac{d}{dz}+\alpha_1\right)\cdots\left(z\frac{d}{dz}+\alpha_p\right)\right\}W+Az^{\lambda}=0 \qquad (3.1)$$

where A and λ are constants and $p\leqslant q+1$, can also be found by means of the above method, in terms of generalized hypergeometric functions (of the first and second kind) [7]. When λ, $\lambda+\beta_s-1\neq 0,-1,\ldots$, then a particular solution of (3.1) is given by

$$W^{(0)}=-\frac{Az^{\lambda}}{\lambda(\lambda+\beta_1-1)\cdots(\lambda+\beta_q-1)}\times \qquad (3.2)$$

$$\times\ {}_{p+1}F_{q+1}(\alpha_1+\lambda,\ldots,\alpha_p+\lambda,1;\beta_1+\lambda,\ldots,\beta_q+\lambda,1+\lambda;z)$$

When $\lambda=-m$ $(m=0,1,\ldots)$, then the generalized hypergeometric function of the second kind

$$W^{(0)}=-\frac{A\,[\alpha_1-m]_m\cdots[\alpha_p-m]_m}{[-m]_m\,[\beta_1-m]_m\cdots[\beta_q-m]_m\,(\beta_1-m-1)\ldots(\beta_q-m-1)}\times$$

$$\times{}_{p+1}\Phi_{q+1}(1+m,\alpha_1,\ldots,\alpha_p;\ 1+m,\beta_1,\ldots,\beta_q;\ z) \qquad (3.3)$$

$\alpha_r\neq 1,2,\ldots,m;\ r=1,2,\ldots,p;\ \beta_s\neq m+1,m,\ldots,1,0,-1,-2,\ldots;\ s=1,2,\ldots,q)$
is a particular solution of (3.1).

Solution (3.3) loses its sense of one, two etc. of the parameters $\beta_s (s=1,\ldots,q)$ are equal to integers $1,\ldots,m+1$. In this case the expression for a particular solution of (3.1) contains a generalized hypergeometric function of the second kind of the form (2.5), (2.7) etc., namely for $\lambda = -k,\ \beta_1 = 1 + m\ (k,\ m = 0,\ 1\ \ldots;\ k \geqslant m)$

$$W^{(0)} = (-1)^{k-m+1} \frac{[\alpha_1 - k]_k \ldots [\alpha_p - k]_k A}{2m!\,(k-m)!\,[-k]_k\,[\beta_2 - k - 1]_{k+1} \ldots [\beta_q - k - 1]_{k+1}} \times \tag{3.4}$$

$$\times\, {}_{p+1}\Phi^{(2)}_{q+1}(1 + k,\ \alpha_1, \ldots, \alpha_p;\ 1 + m,\ 1 + k,\ \beta_2, \ldots, \beta_q;\ z)$$

$(\alpha_r \neq 1, 2, \ldots, k;\ r = 1, 2, \ldots, p;\ \beta_s \neq k + 1,\ k, \ldots, 1, 0, -1, -2, \ldots;\ s = 2, 3, \ldots, q$

for $\lambda = -l, \beta_1 = 1 + m, \beta_2 = 1 + k\ (l, k, m = 0, 1, 2, \ldots; l \geqslant k \geqslant m)$

$$W^{(0)} = (-1)^{k+m+1} \frac{[\alpha_1 - l]_l \ldots [\alpha_p - l]_l A}{6m!\,k!\,(l-m)!\,(l-k)!\,[-l]_l\,[\beta_3 - l - 1]_{l+1} \ldots [\beta_q - l - 1]_{l+1}} \times$$

$$\times_{p+1}\Phi^{(3)}_{q+1}(1 + l,\ \alpha_1, \ldots, \alpha_p;\ 1 + m,\ 1 + k,\ 1 + l,\ \beta_3, \ldots, \beta_q;\ z) \tag{3.5}$$

$(\alpha_r \neq 1, 2, \ldots, l;\ r = 1, 2, \ldots, p;\ \beta_s \neq l + l, \ldots, 1, 0, -1, -2, \ldots;\ s = 3,4, \ldots, q)$

etc.

If $\lambda = 1, 2, \ldots$ then, choosing the linear combination of (3.2) and first of the solutions (2.2) as a particular solution of (3.1), we obtain it in the form of a polynomial in z.

If $\lambda = -m$ ($m = 0, 1, \ldots$) and $\beta_1 = 1 + k$ ($k = 1, 2, \ldots; k > m$), then, constructing a definite linear combination of solutions (3.3) and (2.4), we can find a particular solution of a inhomogeneous Eq. (3.1) in the form of a polynomial in $1/z$.

Similar solutions are obtained in cases corresponding to solutions (3.4) and (3.5) if we put, respectively,

$$\lambda = -k, \quad \beta_1 = 1 + m, \quad \beta_2 = 1 + l \quad (l > k \geqslant m)$$

$$\lambda = -l, \quad \beta_1 = 1 + m, \quad \beta_2 = 1 + k, \quad \beta_3 = l \quad (l_1 > l \geqslant k \geqslant m)$$

4. Some basic properties of generalized hypergeometric functions. **In the papers, [2, 3, 4 and 7] some properties of $\Phi(a, b; c; z)$ are generalized for the functions ${}_p\Phi_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$. Introducing a function ${}_p\Psi_q(\alpha_1, \ldots, \alpha_p; \beta_p, \ldots, \beta_q; z)$ into functional relations for ${}_p\Phi_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$ according to the Formula (0.2), we obtain the following basic functional relations.**

A differential formula

$$\frac{d}{dz}\,{}_p\Psi_q = \frac{\alpha_1 \ldots \alpha_p}{(1 + m)\,\beta_2 \ldots \beta_q}\,{}_p\Psi_q(\alpha_1 + 1, \ldots, \alpha_p + 1;\ 2 + m,\ \beta_2 + 1, \ldots, \beta_q + 1;\ z) \tag{4.1}$$

and recurrent relations

$$\left(z\frac{d}{dz} + \alpha_n\right){}_p\Psi_q = \alpha_n\,{}_p\Psi_q(\alpha_n + 1) \tag{4.2}$$

$$\left(z\frac{d}{dz} + \beta_m - 1\right){}_p\Psi_q = (\beta_m - 1)\,{}_p\Psi_q(\beta_m - 1) \tag{4.3}$$

$$z\frac{d}{dz}\,{}_p\Psi_q = \frac{\alpha_n(\beta_m - 1)}{\beta_m - 1 - \alpha_n}\left[{}_p\Psi_q(\alpha_n + 1) - {}_p\Psi_q(\beta_m - 1)\right] \tag{4.4}$$

$$(\alpha_n - \beta_m + 1)\,{}_p\Psi_q = \alpha_n\,{}_p\Psi_q(\alpha_n + 1) - (\beta_m - 1)\,{}_p\Psi_q(\beta_m - 1) \tag{4.5}$$

$$(n = 1, 2, \ldots, p;\ m = 1, 2, \ldots, q$$

where

$${}_p\Psi_q = {}_p\Psi_q(\alpha_1, \ldots, \alpha_p;\ 1 + m,\ \beta_2, \ldots, \beta_q;\ z)$$

$${}_p\Psi_q(\alpha_n + 1) = {}_p\Psi_q(\alpha_1, \ldots, \alpha_n + 1, \ldots, \alpha_p;\ 1 + m,\ \beta_2, \ldots, \beta_q;\ z) \tag{4.6}$$

$${}_p\Psi_q(\beta_m - 1) = {}_p\Psi_q(\alpha_1, \ldots, \alpha_p;\ 1 + m,\ \beta_2, \ldots, \beta_m - 1, \ldots, \beta_q;\ z)$$

Relations (4.1) to (4.5) formally coincide with the corresponding relations for ${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$.

Integral representation for the function ${}_p\Psi_q(\alpha_1,\ldots,\alpha_p;\ 1+m,\ \beta_2,\ldots,\ \beta_q;\ z)$ in the form of the Mellin-Burns integral is analogous to the Burns's integral representation for the function ${}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\ \beta_q)$[15] and has the form [9 and 6]

$$(-1)^{m-1}\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{m!\,\Gamma(\beta_2)\ldots\Gamma(\beta_q)}\,{}_p\Psi_q(\alpha_1,\ldots,\alpha_p;\ 1+m,\ \beta_2,\ldots,\beta_q;\ z)=$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\alpha_1+s)\ldots\Gamma(\alpha_p+s)\Gamma(-m-s)\Gamma(-s)}{\Gamma(\beta_2+s)\ldots\Gamma(\beta_q+s)}\,z^s\,ds \qquad (4.7)$$

$$\left(|\arg z|<\min\left(\pi,\ \frac{4-\mu}{2}\pi\right),\quad \mu=q+1-p,\quad |z|<1 \text{ when } p=q+1\right)$$

Here the path of integration is curved so that the poles of the integrand function at the points $s=-\alpha_1-n,\ldots,-\alpha_p-n$ $(n=0,1,2,\ldots)$ remain to the left of the path and at the $s=-m,-m,-(m-1),\ldots,-1,0,1,2,\ldots$, on the right-hand side of the path. Validity of (4.7) can be easily confirmed by calculating the integral as a sum of residues of the integrand at the poles $\Gamma(-m-s)\Gamma(-s)$.

When $p=q+1$, then a well known method [16] allows us to obtain, from the integral representation (4.7), the following formula for analytic continuation of the function

$${}_{q+1}\Psi_q(\alpha_1,\ldots,\alpha_{q+1};\ 1+m,\ \beta_2,\ldots,\ \beta_q;\ z)$$

into the region $|z|>1$

$$(-1)^{m-1}\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_{q+1})}{m!\,\Gamma(\beta_2)\ldots\Gamma(\beta_q)}\,{}_{q+1}\Psi_q(\alpha_1,\ldots,\alpha_{q+1};\ 1+m,\beta_2,\ldots,\beta_q;\ z)=$$

$$=\sum_{t=1}^{q+1}\frac{\Gamma(\alpha_t)\Gamma(\alpha_t-m)\Gamma(\alpha_1-\alpha_t)\ldots\Gamma(\alpha_{t-1}-\alpha_t)\Gamma(\alpha_{t+1}-\alpha_t)\ldots\Gamma(\alpha_{q+1}-\alpha_t)}{\Gamma(\beta_2-\alpha_t)\ldots\Gamma(\beta_q-\alpha_t)}\times \qquad (4.8)$$

$$\times\, z^{-\alpha_t}\,{}_{q+1}F_q(\alpha_t,\ \alpha_t-m,\ 1+\alpha_t-\beta_2,\ldots,1+\alpha_t-\beta_q,$$

$$1+\alpha_t-\alpha_1,\ldots,1+\alpha_t-\alpha_{t-1},\ 1+\alpha_t-\alpha_{t+1},\ldots,1+\alpha_t-\alpha_{q+1};\ z^{-1})$$

where $|\arg z|<\pi$ and $|z|>1$.

5. Some basic functional relationships for generalized hypergeometric functions. Direct differentiation of the power series (2.5) yields the following Formulas

$$\frac{d}{dz}\,{}_p\Phi_q^{(2)}(\alpha_1,\ldots,\alpha_p;\ 1+m,\ 1+k,\ldots,\beta_q;\ z)= \qquad (5.1)$$

$$=\frac{\alpha_1\ldots\alpha_p}{(1+m)(1+k)\ldots\beta_q}[{}_p\Phi_q^{(2)}(\alpha_1+1,\ldots,\alpha_p+1;\ 2+m,2+k,\ldots,\beta_q+1;\ z)+$$

$$+2A_0\,{}_p\Phi_q(\alpha_1+1,\ldots,\alpha_p+1;\ 2+m,\ 2+k,\ldots,\beta_q+1;\ z)+$$

$$+(A_0^2-B_0)\,{}_pF_q(\alpha_1+1,\ldots,\alpha_p+1;\ 2+m,2+k,\ldots,\beta_q+1;\ z)]$$

$$A_0=\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_p}-\frac{1}{1+m}-\frac{1}{1+k}-\cdots-\frac{1}{\beta_q}$$

$$B_0=\frac{1}{\alpha_1^2}+\cdots+\frac{1}{\alpha_p^2}-\frac{1}{(1+m)^2}-\frac{1}{(1+k)^2}-\cdots-\frac{1}{\beta_q^2}$$

$$\frac{d}{dz}\,z^\lambda\,{}_p\Phi_q^{(2)}(\alpha_1,\ldots,\alpha_p;\ 1+m,\ 1+k,\ldots,\beta_q;\ z)=$$

$$=\lambda z^{\lambda-1}\Big[{}_{p+1}\Phi_{q+1}^{(2)}(\alpha_1,\ldots,\alpha_p,\ \lambda+1;\ 1+m,\ 1+k,\ldots,\beta_q,\ \lambda;\ z)+$$

$$+\frac{2}{\lambda}\,{}_{p+1}\Phi_{q+1}(\alpha_1\ldots,\alpha_p,\ \lambda+1;\ 1+m,\ 1+k,\ldots,\beta_q,\ \lambda;\ z)\Big] \qquad (5.2)$$

Assuming that $\lambda = \alpha_n$ $(n + 1,\dots, p)$ or $\lambda = \beta_m - 1$ $(m = 1,\dots, q)$ we obtain, from (5.2), the following recurrent relations

$$\left(z\frac{d}{dz} + \alpha_n\right) {}_p\Phi_q^{(2)} = \alpha_n\, {}_p\Phi_q^{(2)}(\alpha_n + 1) + 2\,{}_p\Phi_q(\alpha_n + 1) \tag{5.3}$$

$$\left(z\frac{d}{dz} + \beta_m - 1\right) {}_p\Phi_q^{(2)} = (\beta_m - 1)\, {}_p\Phi_q^{(2)}(\beta_m - 1) + 2\,{}_p\Phi_q(\beta_m - 1) \tag{5.4}$$

$$z\frac{d}{dz}\, {}_p\Phi_q^{(2)} = \frac{\alpha_n(\beta_m - 1)}{\beta_m - 1 - \alpha_n}\Big[{}_p\Phi_q^{(2)}(\alpha_n + 1) - {}_p\Phi_q^{(2)}(\beta_m - 1) + \\ + \frac{2}{\alpha_n}\,{}_p\Phi_q(\alpha_n + 1) - \frac{2}{\beta_m - 1}\,{}_p\Phi_q(\beta_m - 1)\Big] \tag{5.5}$$

$$(\alpha_n - \beta_m + 1)\,{}_p\Phi_q^{(2)} = \alpha_n\,{}_p\Phi_q^{(2)}(\alpha_n + 1) - (\beta_m - 1)\,{}_p\Phi_q^{(2)}(\beta_m - 1) + \\ + 2\,{}_p\Phi_q(\alpha_n + 1) - 2\,{}_p\Phi_q(\beta_m - 1) \tag{5.6}$$

where the notation is similar to that in (4.6). Similar differential and recurrent relations can be derived for ${}_p\Phi_q^{3}(\alpha_1,\dots,\alpha_p;\ \beta_1,\dots,\beta_q;\ z)$.

6. Application of hypergeometric functions of the second kind to the theory of plates and shells. We shall now consider some examples using the following notation: h is the thickness of a shell, r is the radius of a circle parallel to the mean surface of the shell, R is the radius of curvature of a spherical shell, φ is the angle made by the normal to the mean surface and the axis of the shell, α is the angle made by the meridian and the axis of a conical shell, N_r and N_θ are meridional and tangential stresses, $\varkappa_r$, and $\varkappa_\theta$ are the changes of curvature, E and ν are the elasticity modulus and the Poisson's ratio, l is the meridional coordinate for a conical shell, r and θ are polar coordinates for a plate, w is flexure of a plate, $T - T_0$ is the change of temperature and α_t is the linear coefficient of thermal expansion.

1°. ***Axisymmetric deformation of a spherical shell.*** The homogeneous equation of this problem, in terms of a complex function taken as a solution,

$$N = N_r + k_0\varkappa_\theta \quad (k_0 = (\nu \pm i\mu)\,Eh^3/c_0^2R, \quad \mu = \sqrt{c_0^2R^2/h^2 - \nu^2}, \quad c_0^2 = 12(1 - \nu^2)) \tag{6.1}$$

is transformed, on introducing a new function W and another variable ξ given by

$$N = \cos\varphi W, \quad \xi = \sin^2(\varphi/2) \tag{6.2}$$

into (1.1), in which

$$z = \xi; \quad a, b = {}^3/_2 \pm \delta, \quad \delta = {}^1/_2\sqrt{5 \mp 4i\mu}; \quad c = 2$$

Using particular solutions (1.2) and (0.1), we obtain the following general solution

$$W = C_1 F({}^3/_2 + \delta,\ {}^3/_2 - \delta;\ 2;\ \xi) + C_2\Psi({}^3/_2 + \delta,\ {}^3/_2 - \delta;\ 2;\ \xi) \tag{6.3}$$

where C_1 and C_2 are complex constants of integration. Applying, subsequently, a well known transformation formula for $F(a, b; c; z)$, Formula (1.5) and relation (1.7) for $\Psi(a, b; c; z)$ and passing to new constants of integration, we obtain

$$W = C_1'(1 - \xi)^{-1} F({}^1/_2 + \delta,\ {}^1/_2 - \delta;\ 2;\ \xi) + C_2'\,\xi^{-1} F({}^1/_2 + \delta,\ {}^1/_2 - \delta;\ 2;\ 1 - \xi) \tag{6.4}$$

When the tables of solutions were computed covering all stresses, moments and displacements, only four infinite series defining functions.

$$\operatorname{Re} F({}^1/_2 + \delta, {}^1/_2 - \delta;\ c;\ \xi), \quad \operatorname{Im} F({}^1/_2 + \delta,\ {}^1/_2 - \delta;\ c;\ \xi) \quad (c = 2,3).$$

were summed.

2°. ***Axisymmetric deformation of a conical shell of linearly varying thickness.***

$$h = h_0(1 - x) \qquad x = l/l_0 \qquad (h_0,\ l_0 = \text{const};\ 0 \leqslant x < 1,\ 0 \geqslant x > -1) \tag{6.5}$$

The homogeneous equation of this problem in terms of a complex function

$$N = N_r + k_0(1 - x)^2\varkappa_\theta$$

$$\left(k_0 = (-1 + \nu \pm i\rho)\frac{Eh_0^3}{c_0^2 l_0 \operatorname{ctg}\alpha}, \quad \rho = \left(c_0^2\frac{l_0^2}{h_0^2}\operatorname{ctg}^2\alpha - (1 - \nu)^2\right)^{1/2}\right) \tag{6.6}$$

is given by (1.1) in which

$$z = x; \quad W = N; \quad a, b = {}^1/_2 \pm \delta, \quad \delta = \sqrt{{}^9/_4 - 2\nu \mp i\rho}; \quad c = 3$$

The general solution for N has the form

$$N = C_1 F({}^1/_2 + \delta,\ {}^1/_2 - \delta;\ 3;\ x) + C_2\Psi({}^1/_2 + \delta,\ {}^1/_2 - \delta;\ 3;\ x) \tag{6.7}$$

and it can be used whether the thickness of the shell decreases ($0 \leqslant x < 1$) or increases ($0 \geqslant x >> -1$) in the direction of the outer contour.

When $-1 \geqslant x > -\infty$, we obtain a solution by applyin g analytic continuation formulas to functions $F(a, b; c; x)$ and $\Psi(a, b; c; x)$.

When $0 \leqslant x < 1$, then Formulas (1.7) and (1.5) should be used to obtain the second particular solution (with a constant C_2) in terms of $F(\frac{1}{2} + \delta, \frac{1}{2} - \delta; 3; x)$ and $F(\frac{1}{2} + \delta, \frac{1}{2} - \delta; 3; 1 - x)$. Subsequent separation of (6.7) into a real and imaginary part and computation of stresses, moments and displacements in terms of functional relations for $F(a, b; c; x)$, yields expressions for these magnitudes in terms of eight functions [4] Re $F(\frac{1}{2} + \delta, \frac{1}{2} - \delta; c; x)$ and Im $F(\frac{1}{2} + \delta, \frac{1}{2} - \delta; c; x)$ $(c = 1, 2, 3, 4)$.

Only four series defining functions at $c = 3$ and 4 are summed. The remaining functions are obtained by means of recurrence formulas.

3°. ***Cyclically symmetric bending of a circular plate of radially varying cylindrical rigidity***

$$D = D_0(1 - x), \quad x = (r/r_0)^{\varkappa_0} \quad (r_0, D_0 = \text{const}; \ \alpha_0 = {}^2/m; \ m = 2, 3, \ldots) \tag{6.8}$$

under the action of contour and surface loads, the intensity is given by

$$q = q\, r^j \cos k\theta \qquad (q = \text{const}; \ j = 0, 1, \ldots; \ k = 2, 3, \ldots) \tag{6.9}$$

On substitution of a function W according to

$$w = x^{(2+k)/\alpha_0} W \cos k\theta \tag{6.10}$$

the problem reduces to integration of a differential Eq. (3.1), in which [3]

$$z = x; \qquad p = 4, \quad q = 3; \qquad \beta_1 = 1 + m, \quad \beta_2 = 1 + mk$$

$$\beta_3 = 1 + m(1 + k); \qquad \lambda = \frac{(j + 2 - k)\, m}{2} - 1, \qquad A = -\frac{16 q_j r_0^{j+4}}{D_0 m^4}$$

The parameters α_n $(n = 1, 2, 3, 4)$ are roots of Eq.

$$\alpha^4 - A_1\alpha^3 + A_2\alpha^2 - A_3\alpha + A_4 = 0$$

$$A_1 = 4(k + 1)/\alpha_0 + 2, \qquad A_2 = [4k^2 + 6k(\alpha_0 + 2) + \alpha_0^2 + (7 + \nu)\alpha_0 + 4]/\alpha_0^2$$

$$A_3 = [4k^2(\alpha_0 + 2) + 2k(\alpha_0^2 + 7\alpha_0 + \nu\alpha_0 + 4) + \alpha_0^2(3 + \nu) + 2\alpha_0(3 + \nu)]/\alpha_0^3$$

$$A_4 = [k^2(\alpha_0 - \nu\alpha_0 + 6 + 2\nu) + k(3 + \nu)(\alpha_0 + 2) + 2\alpha_0(1 + \nu)] / \alpha_0^3$$

With the results of Sections 2 and 3 and by substitution (6.10), we obtain a general solution of the considered problem in the form

$$\begin{aligned} w = x^{1/2(2+k)m} \{ & C_1\, {}_4F_3(\alpha_1, \ldots, \alpha_4; 1 + m, 1 + mk, 1 + m + mk; x) + \\ & + C_2\, {}_4\Psi_3(\alpha_1, \ldots, \alpha_4; 1 + m, 1 + mk, 1 + m + mk; x) + \\ & + C_3\, {}_4\Phi_3^{(2)}(\alpha_1, \ldots, \alpha_4; 1 + m, 1 + mk, 1 + m + mk; x) + \\ & + C_4\, {}_4\Phi_3^{(3)}(\alpha_1, \ldots, \alpha_4; 1 + m, 1 + mk, 1 + m + mk; x) + W^{(q)}\} \cos k\theta \end{aligned} \tag{6.11}$$

where C_n $(n = 1, 2, 3, 4)$ are real constants of integration.

The particular solution $W^{(q)}$ of a nonhomogeneous Eq. (3.1) when λ is not equal to an integer, has the form

$$\begin{aligned} W^{(q)} = & -\frac{A x^{1/2(j+2-k)m-1}}{\lambda(\lambda + m)(\lambda + mk)(\lambda + m + mk)} \times \\ & \times {}_5F_4(\alpha_1 - 1 + {}^1/_2(j + 2 - k)m, \ \ldots, \ \alpha_4 - 1 + {}^1/_2(j + 2 - k)m, \ 1 \\ & {}^1/_2(j + 2 - k)m, \ {}^1/_2(j + 2 + k)m, \ {}^1/_2(j + 4 - k)m, \ {}^1/_2(j + 4 + k)m, \ x) \end{aligned} \tag{6.12}$$

When $\lambda = 1, 2, \ldots$ or $\lambda = -\mu$ $(\mu = 0, 1, \ldots; m \leqslant \mu < mk)$, the particular solutions $W^{(q)}$ are obtained in terms of polynomials in x and $1/x$ respectively. Bending moments are computed with help of the formulas of differentiation of generalized hypergeometric functions given in Sections 4 and 5.

4°. ***Cyclically symmetric thermoelastic deformation of a sloping conical shell when tensile and bending deformations are purely thermal.***

$$\varepsilon_T = \frac{1}{h}\int_{-1/2 h}^{1/2 h} \alpha_T(T - T_0)\, d\zeta = \varepsilon_j r^j \cos k\theta, \quad \varkappa_T = \frac{12}{h^3}\int_{-1/2 h}^{1/2 h} \alpha_T(T - T_0)\zeta\, d\zeta = \varkappa_j r^j \cos k\theta$$

$$(\varepsilon_j, \ \varkappa_j = \text{const}; \ j = 0, 1, \ldots; \ k = 2, 3, \ldots) \tag{6.13}$$

Solution of this problem can, after separation of variables and certain substitutions, be reduced to Eq. (3.1) in which [4]

$$z = a_0 r \quad (a_0 = \pm\, i c_0 \varphi / h, \quad c_0 = \sqrt{12(1-\nu^2)}; \qquad p = 2, \quad q = 3$$

$$\alpha_1 = 1 + k, \quad \alpha_2 = 2 + k, \quad \beta_1 = 2, \quad \beta_2 = 1 + 2k, \quad \beta_3 = 2 + 2k, \quad \lambda = j - k$$

$$1 = -\frac{iEh}{a_0^{\,j}}\left[i\varepsilon_j + \frac{(1+\nu)h}{c_0}\varkappa_j\right](j+k)(j+k+1)(j-k)(j-k+1)$$

Function W is related to the complex function

$$N = N_r + N_\theta + k_0(\varkappa_r + \varkappa_\theta) \qquad (k_0 = \pm\, iEh^2/c_0) \tag{6.14}$$

which is a solution of this problem, by the following Formula:

$$N = z^k W \cos k\theta \tag{6.15}$$

The general solution for the function N has the form

$$\begin{aligned} N = [&C_1 z^k\, {}_2F_3(1+k,\ 2+k;\ 2,\ 1+2k,\ 2+2k;\ z) + \\ &+ C_2 z^k\, {}_2\Psi_3(1+k,\ 2+k;\ \ 2,\ 1+2k,\ 2+2k;\ z) + \\ &+ C_3 z^{-k}\, {}_2F_3(1-k,\ 2-k;\ 2,\ 1-2k,\ \ 2-2k;\ z) + \\ &+ C_4 z^{-k}\, {}_2\Phi_3(1-k,\ 2-k,\ 2,\ 1-2k,\ 2-2k;\ z) + N^{(T)}]\cos k\theta \end{aligned} \tag{6.16}$$

where C_n ($n = 1, 2, 3, 4$) are complex constants and function $N^{(T)}$ has the form

$$\begin{aligned} N^{(T)} = &\frac{A\,[j+1]_{k-j}\,[j+2]_{k-j}}{2(k-j-1)!\,(k-j)!\,[j+k]_{1+k-j}\,[j+k+1]_{1+k-j}}\, z^k \times \\ &\times {}_3\Phi_4^{(2)}(k-j+1,\ 1+k;\ 2+k;\ 2,\ k-j+1,\ 1+2k,\ 2+2k;\ z) \end{aligned} \tag{6.17}$$

when $j = 0, 1, 2, \ldots, k-2$.

When $j = k-1, k, \ldots$, the function $N^{(T)}$ can be obtained in the form of polynomials [4].

Having obtained a solution for N, we can find solutions for remaining complex stresses and moments using the well known formulas [4] together with the formulas for differentiation of generalized hypergeometric functions, given in Sections 4 and 5.

In the third and fourth particular solutions (with integration constants C_3 and C_4) hypergeometric functions reduce to elementary functions.

BIBLIOGRAPHY

1. Kovalenko, A.D., Hypergeometric functions of the second kind, applied to some problems of the theory of elasticity. Sb. tr. Inst. stroit. mekh. Akad. Nauk USSR, No. 12, 1950.
2. Kovalenko, A.D., Plates and Shells in Turbine Rotors. Kiev, Izd. Akad. Nauk USSR, 1955.
3. Kovalenko, A.D., Circular Plates of Variable Thickness. M., Fizmatgiz, 1959.
4. Kovalenko, A.D., Grigorenko, Ia.M. and Il'in, L.A., Theory of Thin Conical Shells and its Application in Mechanical Engineering. Kiev, Izd. Akad. Nauk USSR, 1963.
5. Nörlund, N.E., The logarithmic solutions of the hypergeometric equation. Dan. Yid. Selsk., Mat. Fys. Skr., Bd. 2, No. 5, 1963.
6. Beitman, G. and Erdeii, A., Higher Transcendental Functions. Hypergeometric Function. Legendre Function. M. (Russian translation) published by "Nauka", 1965.
7. Kovalenko, A.D., On generalisation of Lommel's solutions. Dopovidi Akad. Nauk USSR, No. 4, 1964.
8. Grigorenko, Ia.M., Recurrent formulas in logarithmic solutions of a problem of flexure of circular plates. Prikl. mekhanika, Vol. 3, No. 4, 1957.
9. Vovkodav, I.P., Integral representations of generalised hypergeometric functions of the second kind. Dopovidi Akad. Nauk USSR, No. 1., 1966,
10. Vovkodav, I.F., On logarithmic particular solutions of the equation of cyclic deformation of a sloping conical shell. Prokl. mekhanika, Vol. 2, No. 12, 1966.
11. Gauss, C.F., Werke Vol. 3, Göttingen, 1876.
12. Pochhammer, L., Uber die Differentialgleichung der allgemeineren hypergeometrischen Reihe mit zwei endlichen singulären Punkten. J. reine angew. Math., Bd. 102, 1888
13. Bailey, W.N., Generalized hypergeometric series, Cambridge tracts in mathemathics and mathematical physics. Cambridge, No. 32, 1935.

14. Ains, E.L., Ordinary Differential Equations. Khar'kov, DNTVU, 1939.
15. Barnes, E.W., The asymptotic expansion of integral functions defined by generalized hypergeometric series. Proc., London, Math. Soc., Vol. 2, No. 5, 1907.
16. Whittaker, E.T. and Watson, N.G., Course of Modern Analysis, p. 2, (Russian Translation) Fizmatgiz, 1963.

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ON THE DETERMINATION OF KINETIC STRESS FUNCTIONS IN ELASTODYNAMICS PROBLEMS

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The purpose of the paper is the development of a new method of solving dynamic problems of elasticity theory by introducing kinetic stress functions [1 to 3]. Equations which the kinetic stress functions satisfy are presented here, and the form of the general solution of these equations is found.

Let us consider the square of a line element in some Riemann space, which we shall designate as generating:

$$ds^2 = [1 + \varepsilon\varphi_{kk}(x^j, t)]\, dx^k dx^k - c^2 [1 + \varepsilon\varphi_4(x^j,t)]\, dt^2 \quad (k, j = 1, 2, 3) \qquad (1)$$

where ε is an arbitrary small parameterm c^2 a constant to be determined, $\varphi_{kk}(x^j, t) = \varphi_k(x^1, x^2, x^3, t)$ the kinetic stress functions. It is seen from (1) that for $\varepsilon = 0$ the Riemann space degenerates into a Euclidean space. We assume that this Euclidean space contains the continuum being studied. Functional derivatives of the components of the fundamental metric tensor of the generating Riemann space define the kinetic stress tensor as $\varepsilon \to 0$.

We assume that the energy momentum tensor is proportional to the functional derivative of the fundamental geometric invariant [4]. Let us set

$$T^{\mu\nu} = \varepsilon^{-1}(R^{\mu\nu} - {}^1/_2\, g^{\mu\nu} R) \qquad (2)$$

where $T^{\mu\nu}$ is the energy-momentum tensor; the remaining notation is standard.

As a result of passing to the limit as $\varepsilon \to 0$ we obtain a general solution of the equations of motion of a continuum element from (2) [2]:

$$\sigma_{ii} - \rho v_{ii}^2 = \frac{1}{2}\left[\frac{\partial^2(\varphi_j + \varphi_4)}{\partial x^k \partial x^k} + \frac{\partial^2(\varphi_k + \varphi_4)}{\partial x^i \partial x^j} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\varphi_k + \varphi_j)\right] \qquad (3)$$

$$\sigma_{kj} - \rho v_k v_j = -\frac{1}{2}\frac{\partial^2(\varphi_i + \varphi_4)}{\partial x^k \partial x^k} \qquad (4)$$

$$\rho v^i = -\frac{1}{2c^2}\frac{\partial^2(\varphi_k + \varphi)}{\partial x^i \partial t} \qquad (5)$$

$$\rho = \frac{1}{2c^2}\left[\frac{\partial^2(\varphi_3 + \varphi_2)}{\partial x^1 \partial x^1} + \frac{\partial^2(\varphi_1 + \varphi_3)}{\partial x^2 \partial x^2} + \frac{\partial^2(\varphi_1 + \varphi_2)}{\partial x^3 \partial x^3}\right] \qquad (6)$$

Here σ_{ik} is the stress tensor, **v** the velocity of a continuum element, ρ the density. The indices i, k, j generate a cyclic permutation of the numbers 1, 2, 3. We henceforth neglect nonlinear terms in the components of the three-dimensional portion of the kinetic stress tensor in Expressions (3) to (6). The generality of (3) to (6) results, in particular, from the pos-